Int J Theor Phys (2010) 49: 979–987
DOI 10.1007/s10773-010-0277-6

Exact Solutions of the Klein-Gordon Equation for the Scarf-Type Potential via Nikiforov-Uvarov Method

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Received: 13 November 2009 / Accepted: 9 February 2010 / Published online: 2 March 2010

Abstract In this paper we present the exact solutions of the one-dimensional Klein-Gordon equation for the Scarf-type potential with equal scalar and vector potentials. Exact solutions and corresponding energy eigenvalues equation are obtained using Nikiforov-Uvarov mathematical method for the s-wave bound state. The PT-symmetry and Hermiticity for this potential are also considered. It will be shown that the obtained results of the Scarf-type potential are reduced to the results of the well-known potentials in the special cases.

Keywords Klein-Gordon equation · Scarf potential · Nikiforov-Uvarov

1 Introduction

It is well-known that solutions of the relativistic wave equation play an essential role in the relativistic quantum mechanics for some physical potentials of interest [1–5]. Recently, there has been an increasing interest in finding exact solutions of the Klein-Gordon (KG) equation [6–10]. In the past few years, exact solutions and energy eigenvalues of this equation have been presented for Rosen-Morse type [11, 12], Hulthen [13], Wood-saxon [14, 15], Posch-Teller [16], five-parameter exponential [17, 18], generalized symmetrical double-well [19], ring-shape harmonic oscillator [20], and pseudo harmonic oscillator [21] potentials, etc. In the above cited papers the scalar and vector potentials are almost taken to be equal in the relativistic framework. However, there is almost no explicit expression for the energy eigenvalues.

In the present work, first we briefly review Nikiforov-Uvarov (NU) mathematical method in which a second-order linear differential equation is solved by reducing it to a generalized equation of hyper-geometric type [22]. Then, by using this method we solve the KG equation for Scarf-type potential with equal scalar and vector potentials. The exact solutions and corresponding energy eigenvalues equation are obtained. Special forms of the Scarf-type potential are also discussed and some figures for this potential and energy levels are

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presented. In addition, we investigate the PT-symmetry and Hermiticity for the Scarf-type potential in which P is parity operator and T denotes time-reversal operator acting on the Hilbert space $\mathcal{H} = L^2(R)$. The Hermiticity of a Hamiltonian was supposed to be a necessary condition for the real spectrum until the year 1998 [23–26]. A conjecture has ceased this constraint by introducing the concept of PT-symmetric Hamiltonians [23–26]. In the past few years, theoretical researches on a great variety of non-hermitian Hamiltonians have received an important attention [23–27]; some systems are invariant under a combination of parity and time reversal (PT) transformation which lead to either real (in case of broken PT-symmetry) or complex energy eigenvalues (in case of spontaneously broken PT-symmetry) [23–27]. This property of energy eigenvalues in non-hermitian PT-invariant systems can be related to the pseudo-Hermiticity [28] or anti-unitary symmetry [29] of the corresponding Hamiltonians. In Ref. [30] it has been proposed a new class of non-hermitian Hamiltonians with real spectra which are obtained using pseudo-symmetry. Moreover, in Ref. [31] completeness and orthonormality conditions for eigenstates of such potentials have been proposed. In the study of PT-invariant potentials various techniques have been applied to a great variety of quantum mechanical approaches such as variational methods, numerical techniques, Fourier analysis, semi-classical estimates, quantum field theory and Lie group theoretical approaches [31–34]. In addition, PT-symmetric, non-PT-symmetric and also non-hermitian potentials such as oscillator-type and a variety of potentials within the framework of SUSYQM [35–40] have been studied [41–45].

This paper is organized as follows. In Sect. 2 we briefly review NU mathematical method. Section 3 is devoted to the derivation of the exact solutions of the KG equation and its energy eigenvalues equation for the Scarf-type potential. We investigate the PT-symmetry and Hermiticity, as well as, specific forms of this potential in Sect. 4. The conclusion is given in Sect. 5.

2 Overview of NU Method

NU method is based upon reducing the second order linear differential equation to a generalized equation of hyper-geometric type [22]. This method provides exact solutions in terms of special orthogonal functions as well as corresponding eigenvalues. In both relativistic and non-relativistic quantum mechanics, the wave equation with a given real or complex potential can be solved by this method. In the present work, we use this method for solving the KG equation with equal scalar and vector potentials. By introducing an appropriate coordinate transformation $s = s(x)$, one can rewrite this equation in the following form

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of degree two at most, and $\tilde{\tau}(s)$ is a polynomial of degree one at most [22]. Now, if one takes the following factorization

$$\psi(s) = y_n(s)\phi(s). \tag{2}$$

Equation (1) reduces to a hyper-geometric type equation of the form

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0,$$

where $\tau(s)=\tilde{\tau}(s)+2\pi(s)$ satisfies the condition $\tau'(s)<0$, and $\pi(s)$ is defined as

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+\kappa\sigma(s)} \tag{3}$$

in which κ is a parameter. Determining κ is the essential point in the calculation of $\pi(s)$. It is obtained by setting the discriminant of the square root equal to zero [22]. Therefore, one gets a general quadratic equation for κ. By using

$$\lambda=\kappa+\tilde{\pi}(s)=-n\tilde{\tau}(s)-\frac{n(n-1)}{2}\sigma''(s) \tag{4}$$

the values of κ can be used for the calculation of energy eigenvalues. Polynomial solutions $y_n(s)$ are given by the Rodrigues relation

$$y_n(s)=\frac{B_n}{\rho(s)}\left(\frac{d}{ds}\right)^n[\sigma^n(s)\rho(s)] \tag{5}$$

in which B_n is a normalization constant and $\rho(s)$ is the weight function satisfying

$$\rho(s)=\frac{1}{\sigma(s)}\exp\int\frac{\tau(s)}{\sigma(s)}ds. \tag{6}$$

On the other hand, second part of the wave-function $\phi(s)$ in relation (2) reads [22]

$$\phi(s)=\exp\int\frac{\pi(s)}{\sigma(s)}ds. \tag{7}$$

3 Exact Solutions

The one dimensional KG equation for a spinless particle of rest mass m in the natural units $\hbar=c=1$ can be written as

$$\psi''(x)+[(E-V(x))^2-(m+S(x))^2]\psi(x)=0,$$

where E, $V(x)$ and $S(x)$ are the relativistic energy of the particle, vector and scalar potentials, respectively. Recently, interest in solutions of this equation with equal scalar and vector potentials has been increased [11, 12]. Under assumption $V(x)=S(x)$ the last equation takes the form

$$\psi''(x)+\{(E^2-m^2)+2(E+m)V(x)\}\psi=0. \tag{8}$$

Here we consider the one dimensional Scarf-type potential as

$$V(x)=-V_0\operatorname{sech}_q^2(\alpha x)+V_1\operatorname{sech}_q(\alpha x)\tanh_q(\alpha x), \tag{9}$$

where we have used the following deformed hyperbolic functions [46]

$$\sinh_q(x)=\frac{e^x-qe^{-x}}{2},\quad \cosh_q(x)=\frac{e^x+qe^{-x}}{2},\quad \tanh_q(x)=\frac{\sinh_q(x)}{\cosh_q(x)},$$

$$\coth_q(x)=\frac{\cosh_q(x)}{\sinh_q(x)},\quad \operatorname{sech}_q(x)=\frac{1}{\cosh_q(x)},\quad \operatorname{cosech}_q(x)=\frac{1}{\sinh_q(x)}.$$

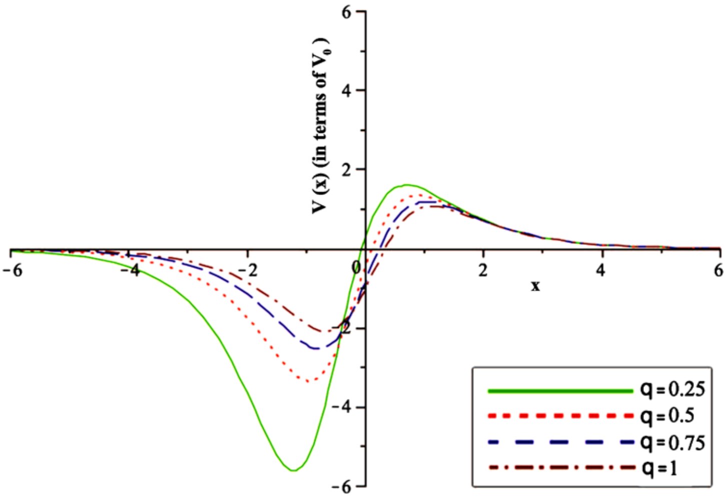

Fig. 1 The variation of the Scarf-type potential in terms of x, for four values of q with $V_1 = \frac{1}{3}V_0$ and $\alpha = 1$

The deformed hyperbolic functions allow us to investigate the effect of the deformation parameter on the energy levels and the corresponding wave functions. Consequently, the present deformed potential is reduced to some standard potential by choosing an appropriate value for the deformation parameter. The Scarf-type potential is transformed into the standard Scarf II potential for $q = 1$, into the Poschl-Teller II and generalized Poschl-Teller potentials for $q = -1$ and into the exponential potential for $q = 0$, apart from a constant shift. By using a coordinate translation transformation, it has been shown that deformed hyperbolic potentials can be transformed into the corresponding non-deformed ones [28]. The deformation parameter has been taken positive and real in Ref. [27], and the energy eigenvalues equation and the corresponding wave function are obtained. Extension to complex parameters has been considered in Refs. [23–26].

Figure 1 shows a schematic representation of the Scarf-type potential in terms of x, for four different values of the deformation parameter q.

By introducing a new variable $s = iq^{-1/2}\sinh_q(\alpha x)$, it is straightforward to show that (8) takes the form

$$\psi''(s) - \frac{s}{1-s^2}\psi'(s) - \frac{1}{(1-s^2)^2}[\epsilon^2 s^2 + i\gamma^2 s + \beta^2 - \epsilon^2]\psi(s) = 0,$$

where, we have used

$$\epsilon^2 = \frac{m^2 - E^2}{\alpha^2}, \qquad \gamma^2 = \frac{2(E+m)V_1}{\alpha^2\sqrt{q}}, \qquad \beta^2 = \frac{2(E+m)V_0}{\alpha^2 q}.$$

By comparing the last equation with (1), we get

$$\tilde{\tau}(s) = -s, \qquad \sigma(s) = 1 - s^2, \qquad \tilde{\sigma}(s) = -[\epsilon^2 s^2 + i\gamma^2 s + \beta^2 - \epsilon^2]. \tag{10}$$

Substituting them into relation (3) leads to

$$\pi(s) = -\frac{s}{2} \mp \frac{1}{2}\begin{cases} as - b & \text{for } \kappa = \epsilon^2 - \frac{1}{4}b^2 + 1/4 \\ as + b & \text{for } \kappa = \epsilon^2 - \frac{1}{4}a^2 + 1/4, \end{cases}$$

where

$$a = -\sqrt{(2\beta^2 + 1/2) + \sqrt{(\beta + 1/4)^2 + 4\gamma^4}},$$
$$b = -\sqrt{(2\beta^2 + 1/2) - \sqrt{(\beta + 1/4)^2 + 4\gamma^4}}. \quad (11)$$

By choosing an appropriate value for κ in $\pi(s)$ which satisfies the condition $\tau'(s) < 0$ one gets

$$\pi(s) = -\frac{1}{2}[(a+1)s + b] \quad \text{for } \kappa = \epsilon^2 - \frac{1}{4}a^2 + \frac{1}{4}.$$

From (6) it can be shown that the weight function $\rho(s)$ is

$$\rho(s) = \left(\frac{1-s}{2}\right)^{\frac{a+b}{2}} \left(\frac{1+s}{2}\right)^{\frac{a-b}{2}}$$

and by substituting $\rho(s)$ into the Rodrigues relation (5) one gets

$$y_n = B_n \left(\frac{1-s}{2}\right)^{-\frac{a+b}{2}} \left(\frac{1+s}{2}\right)^{-\frac{a-b}{2}} \left(\frac{d}{ds}\right)^n \left[(1-s^2)^n \left(\frac{1-s}{2}\right)^{\frac{a+b}{2}} \left(\frac{1+s}{2}\right)^{\frac{a-b}{2}}\right]$$
$$= P_n^{(\frac{a+b}{2}, \frac{a-b}{2})}(s), \quad (12)$$

where $P_n^{(\frac{a+b}{2}, \frac{a-b}{2})}(s)$ stands for the Jacobi polynomial and $B_n = \frac{(-1)^n}{n!2^n}$ is the normalization constant. The other part of the wave function is simply found from (7) as

$$\phi(s) = \left(\frac{1-s}{2}\right)^{\frac{a+b+1}{4}} \left(\frac{1+s}{2}\right)^{\frac{a-b+1}{4}}.$$

Finally, by multiplying the two parts, the wave function is obtained as follows

$$\psi(s) = \left(\frac{1-s}{2}\right)^{\frac{a+b+1}{4}} \left(\frac{1+s}{2}\right)^{\frac{a-b+1}{4}} P_n^{(\frac{a+b}{2}, \frac{a-b}{2})}(s). \quad (13)$$

The corresponding energy eigenvalues equation is found from (4) as

$$\left\{\frac{V_0\bar{E}}{q} + \frac{\alpha^2}{8} + \left\{\left(\frac{V_0\bar{E}}{2q} + \frac{\alpha^2}{8}\right)^2 + \frac{{V_1}^2\bar{E}^2}{q}\right\}^{1/2}\right\}^{1/2} + \sqrt{\bar{E}(2M - \bar{E})} = (n + 1/2)\alpha, \quad (14)$$

where $\bar{E} = E + M$.

This is the energy eigenvalues equation for the KG equation with equally mixed Scarf-type potentials. The energy levels can be numerically determined for given values of constant coefficients by using a mathematical software like Maple. The variation of the energy levels in terms of V_0 are given in Figs. 2 and 3 for different values of q and n with $V_1 = \frac{1}{3}V_0$ and $\alpha = 1$.

These results are consistent with those given in Ref. [18] which are obtained using the supersymmetric quantum mechanics. In order to check this, one can insert $q \to 1$, $V_0 \to A(A+\alpha) - B^2$ and $V_1 \to B(2A + \alpha)$ in (14) and easily reach to (34) in Ref. [18].

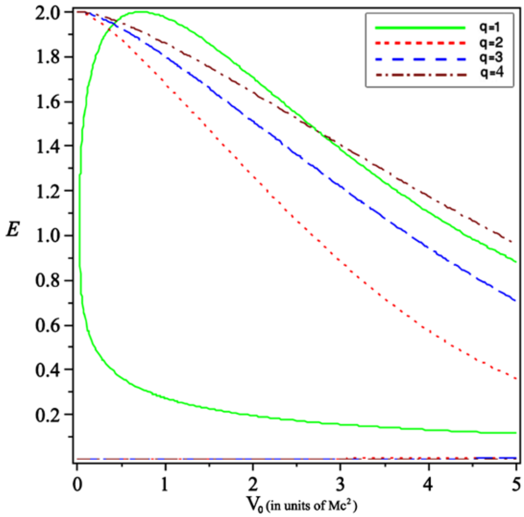

Fig. 2 The variation of the energy in terms of V_0 for the Scarf-type potential in the KG theory for four values of q with $V_1 = \frac{1}{3} V_0$ and $\alpha = 1$

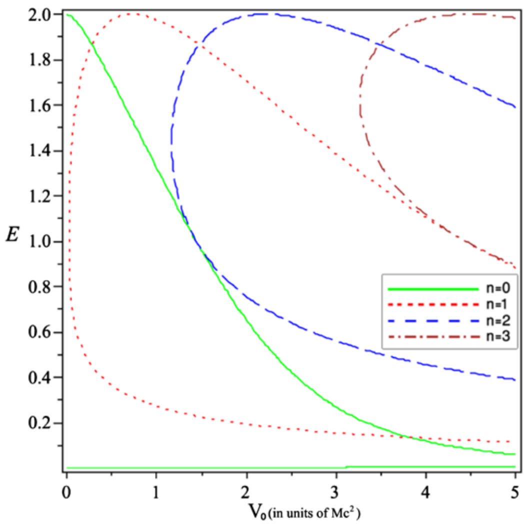

Fig. 3 The variation of the energy in terms of V_0 for the Scarf-type potential in the KG theory for four values of n with $V_1 = \frac{1}{3} V_0$ and $\alpha = 1$

4 Discussion

In this section, first PT-symmetry and the Hermiticity for the Scarf-type potential are considered. Then, by choosing appropriate parameters in this potential we obtain some well-known potentials and discuss their energy equations within the framework of the KG equation.

4.1 PT-symmetry and Non-Hermiticity

A Hamiltonian is said to be PT-symmetric when $[\mathrm{H}, \mathrm{PT}] = 0$, where P is parity operator (space reflection, $x \to -x$), and T denotes time-reversal operator (complex conjugation, $i \to -i$). As one can see from (9), the original potential $V(x)$ is a hermitian function that dose not admit this symmetry. However, we consider the case in which α is a pure imaginary parameter, i.e. $\alpha \to i\alpha$. Under this replacement and after some manipulations, the potential takes the following form

$$V(x) = -4\frac{2V_0 - (1-q^2)V_1 \cos\alpha x - i(1+q^2)V_1 \sin\alpha x}{(1+q^2)\cos 2\alpha x + i(1-q^2)\sin 2\alpha x + 2q}. \tag{15}$$

Clearly this potential is PT-symmetric. The corresponding energy eigenvalues equation of (15) is given by

$$\left\{\frac{V_0\bar{E}}{q} - \frac{\alpha^2}{8} + \left\{\left(\frac{V_0\bar{E}}{2q} - \frac{\alpha^2}{8}\right)^2 + \frac{{V_1}^2\bar{E}^2}{q}\right\}^{1/2}\right\}^{1/2} + \sqrt{\bar{E}(2M-\bar{E})} = i(n+1/2)\alpha. \tag{16}$$

4.2 Non-PT-symmetry and Non-Hermiticity

Another form of the potential is obtained by assuming some of the potential parameters as pure imaginary, that is, $q \to iq$, $\alpha \to i\alpha$ and $V_0 \to iV_0$ and V_1 is replaced by $\sqrt{i}V_1$. This replacements in (9) after some manipulations leads to the following potential

$$V(x) = -4\frac{2iV_0 - \sqrt{i}V_1(e^{\alpha x} - iqe^{-\alpha x})}{(e^{\alpha x} + iqe^{-\alpha x})^2}. \tag{17}$$

The corresponding energy equation of this potential is also found to be (16). Both PT-symmetric potential (15) and non-PT-symmetric potential (17) are non-hermitian. Although the corresponding Hamiltonians are also non-hermitian, it can be numerically shown that some of the energy eigenvalues of (16) are real for given values of constant coefficients, using a mathematical software.

4.3 Scarf II Potential

By choosing $q = 1$ and introducing

$$V_0 = -(B^2 - A^2 - A\alpha) \quad \text{and} \quad V_1 = B(2A+\alpha) \tag{18}$$

the potential (9) transforms to the standard Scarf II potential [47]

$$V(x) = (B^2 - A^2 - A\alpha)\operatorname{sech}^2(\alpha x) + B(2A+\alpha)\operatorname{sech}(\alpha x)\tanh(\alpha x). \tag{19}$$

The corresponding energy equation is given by

$$\left\{\frac{\alpha^2}{8} - (B^2 - A^2 - A\alpha)\bar{E} + \left\{\left(\frac{\alpha^2}{8} - \frac{(B^2 - A^2 - A\alpha)\bar{E}}{2}\right)^2 + B^2(2A+\alpha)^2\bar{E}^2\right\}^{1/2}\right\}^{1/2} + \sqrt{\bar{E}(2M-\bar{E})} = (n+1/2)\alpha. \tag{20}$$

In the second term of potential (19), when $\operatorname{sech}\alpha x$ is replaced by $\operatorname{sech}^2\alpha x$, the resulting potential describes the vibrations of polyatomic molecules [48].

4.4 Poschl-Teller II Potential

By choosing $q=-1$, replacing $\alpha\to 2\alpha$ and introducing

$$V_0=-2[A(A+\alpha)+B(B-\alpha)] \quad \text{and} \quad V_1=2[A(A+\alpha)-B(B-\alpha)], \quad A>B \tag{21}$$

potential (9) transforms to the Poschl-Teller II potential [47]

$$V(x)=-[A(A+\alpha)+B(B-\alpha)]\,\mathrm{sech}^2\,\alpha x+[A(A+\alpha)-B(B-\alpha)]\,\mathrm{cosech}^2\,\alpha x.$$

The corresponding energy equation is given by

$$\left\{\frac{\alpha^2}{2}+2[A(A+\alpha)+B(B-\alpha)]\bar{E}+\left\{\left(\frac{\alpha^2}{2}\right)^2+[A(A+\alpha)+B(B-\alpha)]\bar{E}\right.\right.$$
$$\left.\left.-4[A(A+\alpha)-B(B-\alpha)]^2\bar{E}^2\right\}^{1/2}\right\}^{1/2}+\sqrt{\bar{E}(2M-\bar{E})}=(2n+1)\alpha. \tag{22}$$

4.5 Generalized Poschl-Teller Potential

By setting $q=-1$ and introducing

$$V_0=-(B^2+A^2+A\alpha) \quad \text{and} \quad V_1=-B(2A+\alpha), \quad A<B \tag{23}$$

the potential (9) transforms to the generalized Poschl-Teller potential [47]

$$V(x)=(A^2+A\alpha+B^2)\,\mathrm{cosech}_q^2\,(\alpha x)-B(2A+\alpha)\,\mathrm{cosech}_q\,(\alpha x)\coth_q(\alpha x). \tag{24}$$

Making the parameter replacement (23) in the energy equation (14) leads to

$$\left\{\frac{\alpha^2}{8}+(B^2+A^2+A\alpha)\bar{E}+\left\{\left(\frac{\alpha^2}{8}+\frac{(B^2+A^2+A\alpha)\bar{E}}{2}\right)^2-B^2(2A+\alpha)^2\bar{E}^2\right\}^{1/2}\right\}^{1/2}$$
$$+\sqrt{\bar{E}(2M-\bar{E})}=(n+1/2)\alpha. \tag{25}$$

The energy eigenvalues equations (20), (22) and (25) can be numerically solved for given values of constant coefficients using a mathematical software.

5 Conclusion

We have solved the one dimensional KG equation for the Scarf-type potential with equal scalar and vector potentials for s-wave bound states. The exact solutions and the corresponding energy eigenvalues equation have been found using NU method. In addition, the PT-symmetry was also discussed for this potential. Finally, as the special cases of the Scarf-type potential, standard Scarf II, Poschl-Teller II and generalized Poschl-Teller potentials were considered.

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